Progressive Powered Lenses: the Minkwitz Theorem

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ABSTRACT: Purpose. The Minkwitz theorem, which can be proven to apply to the immediate surface surrounding a line of umbilics, states that astigmatism perpendicular to the line changes twice as quickly as the rate of change of power along the line. Our objective is to test how the Minkwitz theorem applies to the design of progressive addition lenses (PALs).

Method. Our primary investigation of the astigmatism/power rate relationship used Hoya Tact lenses because they have a relatively large central region with horizontal spherical equivalent power contours and vertical astigmatism power contours. Other PALs were used for subsequent analysis. Lenses were measured with a Rotlex Class Plus lens analyzer.

Results. Zone widths in the central region of the Tact lenses exceeded those predicted by the Minkwitz theorem. Above and below this region, zone widths were narrower than predicted. When averaged along the entire corridor, zone widths approximated the Minkwitz theorem. For other PALs, the measured zone widths exceed Minkwitz theorem in the top (distance) and middle (intermediate) corridor but fell short in the lower (near) corridor. Likewise, on average along the entire corridor, they approximate the Minkwitz theorem. Conclusions. Although the Minkwitz theorem must apply exactly to the immediate locale of an umbilic, deviations from Minkwitz can occur within 2 mm of the corridor. Several factors enable enough local deviation from the Minkwitz theorem to "steer" the astigmatism and affect its magnitude in the peripheral portions of a lens. Although the Minkwitz relationship may be altered in some regions of the corridor, there is a global component to the Minkwitz prediction that applies to PALs. On a global level, the gains and losses of astigmatism along the corridor with respect to the Minkwitz prediction have strong tendency to cancel one another. In the end, it appears the unwanted astigmatism associated with a given power change along a given distance can be redistributed but probably not reduced. (Optom Vis Sci 2005;82:1–1)

Key Words: optics, progressive addition lenses, lenses, astigmatism, presbyopia, bifocals

Progressive addition lenses (PALs) are commonly used in the correction of presbyopia and currently account for approximately half of all multifocal sales. PALs are characterized by having a vertex line along which spherical add power increases toward the bottom of the lens (Fig. 1). Flanking the vertex line is unwanted astigmatism that increases in power with greater distance from the vertex line. A corridor with width defined to selected limits of unwanted astigmatism surrounds the vertex line. The rate of power change along the vertex line is greatest in the region of intermediate add power where the zone width is narrowest. For most PALs, there is a relatively large area of stable power for distance viewing in the top of the lens and a somewhat smaller-sized area of stable power for near viewing in the bottom of the lens.

The buildup of astigmatism lateral to the vertex line is a necessary consequence of the power progression. The theoretical basis for the relationship between the unwanted astigmatism and the rate of power progression was addressed by Minkwitz. The Minkwitz theorem states that perpendicular to the vertex line, the surface astigmatism changes twice as quickly as the rate of change of power along the vertex line. This can be mathematically stated as:

$$\frac{\partial P_a}{\partial x} = 2 \frac{\partial P_s}{\partial y}$$

where P_a and P_s are astigmatic and mean spherical power, respectively, x is the distance from the vertex line, and y is distance along the vertex line.

In 1976, Humphrey and Schonhofer also wrote that a surface with change of sphere power along a line will have lateral astigmatism that builds at twice the rate of power change. Humphrey stated that Joseph Weinberg had pointed out this relationship...
many years earlier, but no citation is given. In 1978, Alvarez published a report regarding the design of a variable-focus lens comprised of 2 separate lenses. The combined power of the 2 lenses created variable power when translated with respect to one another along a selected axis. Such a lens pair came to be known as an "Alvarez lens" and was used commercially in the Humphrey Vision Analyzer, which instrument is no longer available. Each of the 2 lenses had progressive power with unwanted astigmatism lateral to the corridor and hence had optics similar to a PAL. There is no evidence that Alvarez was aware of the fact that each of the 2 elements in his 2-lens system behaved like a PAL. Alvarez mentions bifocal lenses but does not mention PALs in his paper.

The analysis in Appendix A provides an alternative approach to the one that Minkwitz used, but confirms the same relationship. It assumes a surface with a constant rate of power change. Surface astigmatism perpendicular to the vertex line is predicted to increase at twice the power rate along the vertex line. The axis of the astigmatism is diagonal. The diagonal nature of the astigmatism enabled development of the Alvarez lenses. Each lens of the Alvarez pair was essentially the same and with a constant power rate along the vertex line. The 2 lenses were rotated 180° with respect to one another and then translated in opposing directions along the vertex line. Hence, the axis of the unwanted astigmatism of each lens was 90° from the other and the astigmatism of each lens canceled the other, a condition that requires diagonal orientation of the axes.

In PALs, the unwanted astigmatism is typically characterized in terms of the total corridor width to a defined magnitude of astigmatism—corridor width is twice that predicted by the Minkwitz relationship because it extends to both sides of the vertex line. The predicted corridor widths for selected astigmatism magnitudes as a function of power rate in the center of the corridor are shown in Figure 2.

The build of astigmatism in a PAL must obey the Minkwitz relationship—at least for some distance orthogonal to the center of the corridor. However, there is wide variance in the distribution of unwanted astigmatism of the various PAL designs currently available, indicating that designers have some flexibility to alter the manner in which the unwanted astigmatism is distributed across the lens. The primary objective of this study is to measure the relationship between power change and astigmatism buildup in PALs and determine how the Minkwitz theorem applies to the design of PALs.

METHODS

The Minkwitz theorem is based on changes in surface curvature. In this study, the effects of the surface curvature changes are studied by measuring the through power of progressive powered ophthalmic lenses with a spherical back surface. The procedures were designed to eliminate the optical effects of the back surface, but small effects such as unintended asphericity or the effects of obliquity of the measuring apparatus are possible confounders.

The Minkwitz theorem predicts that a lens with a constant rate of power change will result in spherical equivalent power contours being horizontal and astigmatism power contours being vertical as shown in Appendix A. This would appear to be the simplest lens design on which to study the Minkwitz relationship, although most current PALs have designs that are considerably more complex than this simple situation. In measuring the contour plots of numerous lenses with progressive power, we have identified a design that has power contours similar to those of the simple progressive lens shown in Appendix A. The power contours of Tact, an occupational progressive lens manufactured by Hoya, are shown in Figure 3.

The Hoya Tact lens is designed as an occupational progressive lens (OPL), i.e., it is intended to be worn for indoor work such as for office work, including computer use. In comparison to general-wear PALs, OPLs have larger intermediate and near zones. The top of the lens contains a far-intermediate viewing zone and sometimes a small distance zone. OPLs also have a longer corridor of power change with a lower power rate and hence lower magnitude of...
unwanted astigmatism. The Hoya Tact was selected for analysis because, even within the OPL category, it has a unique optical design. As shown in Figure 3, in the central portion of the lens, the spherical equivalent power contours are nearly horizontal and the astigmatism contours are nearly vertical, as predicted by the Minkwitz theorem for a lens with constant power rate shown in Appendix A. Tact also has similar changes in zone width in the upper and lower portions of the corridor, i.e., the astigmatism contours flare similarly at the top and bottom. The unwanted astigmatism does not appear to have been "moved" to the lower quadrants like in other PALs. Because of its relatively unique optical properties, we chose to use the Tact lens as the primary test of the application of the Minkwitz theorem to progressive lenses. However, other PALs are also measured as part of this investigation.

Measurement Methods

Lenses were measured with a Rotlex Class Plus lens analyzer that determines lens contour plots with a single measurement. The instrument is basically a moiré deflectometer with a point source rather than a collimated beam. Diverging light from a laser point source, approximately perpendicular to the back surface, is incident directly on the tested lens. The location of the laser point source is variable dependent on the power of the lens, for lenses of plano power (in this study, power ranged from plano to +2.50 D) it is 92 mm; for +10, it is 50 mm. The rays refracted by the lens pass through 2 gratings and form a moiré pattern on a diffusive screen. Proprietary image processing algorithms convert the fringe data to arrays of local wavefront properties, in particular the 2 principal curvatures and axis directions. These arrays are used to calculate 2-dimensional maps of local power, cylinder, and axis of progressive lenses and other phase objects with variable powers. The manufacturer calibrates the instrument to perform directly on the tested lens. The location of the laser point source, approximately perpendicular to the back surface, is incident directly on the tested lens. The location of the laser point source is variable dependent on the power of the lens, for lenses of plano power (in this study, power ranged from plano to +2.50 D) it is 92 mm; for +10, it is 50 mm. The rays refracted by the lens pass through 2 gratings and form a moiré pattern on a diffusive screen. Proprietary image processing algorithms convert the fringe data to arrays of local wavefront properties, in particular the 2 principal curvatures and axis directions. These arrays are used to calculate 2-dimensional maps of local power, cylinder, and axis of progressive lenses and other phase objects with variable powers. The manufacturer calibrates the instrument to perform within ±0.03 D for 5 glass-certified standard lenses.

The Rotlex software averages values within a 4-mm diameter circle, i.e., the power assigned to each location on the lens is essentially an average of the 4-mm diameter area surrounding it. Because the Minkwitz theorem predicts a constant rate of astigmatism build lateral to the umbilicus, the average astigmatism on either edge of the 4-mm circle will equal the magnitude of the astigmatism at the center; therefore, averaging over a 4-mm diameter does not adversely affect the validity of the measurement provided the 4-mm diameter is entirely on one side of the corridor. Therefore, assuming the steady build of astigmatism predicted by Minkwitz, valid measurements of astigmatism can be obtained at 2 mm from the umbilicus and beyond. However, if the rate of astigmatism change within the 4-mm aperture is not constant, then the power assigned to the center will have an associated error. Most of the data in this study are reported in terms of the "zone widths" or the distance between astigmatic contours of equal value on opposite sides of the umbilicus. Therefore, because of the averaging, zone widths of 4 mm or greater can be measured without the measurement aperture, including both sides of the corridor.

PALs were measured with the prism reference line markings (the lens markings that are 34 mm apart and represent the 0–180 line on the lens) appropriately aligned in the instrument and the data file was saved after measurement. For Tact lenses, however, lenses were rotated slightly so that the vertex line and optical corridor were oriented vertically. All measurements taken from the data file were determined with the "DST" mode of the instrument, i.e., all of the cylinder measures on each lens were normalized to an assigned power of zero at the manufacturer-specified reference location. This eliminated the effects of laboratory surfacing variances at the reference location on cylinder measurements. Vertical level on the lens was specified in terms of Y-value in millimeters, with positive values representing lower levels in the lens. The zero Y value was established at the fitting cross of the lens or at the prism reference line if a fitting cross location was not specified (such as for the Hoya Tact lens). Data were analyzed in 1-mm vertical steps along the corridor. At each step, the central sphere power and left and right boundaries (X values) of selected astigmatism magnitudes were recorded. Power rate change, unless indicated otherwise, was calculated for each Y level as the difference in power at the Y levels 1 mm above and below the Y level in question divided by 2. Zone width for each astigmatism level was the difference between the X values of the left and right boundaries.

The astigmatism measurements obtained with the Rotlex were calibrated against those obtained with an automated Humphrey Lensmeter (model 360) at 2-mm increments to 16 mm along the

FIGURE 3.
Spherical equivalent and astigmatism (cylinder) contour plots for +2.00 add Tact.
horizontal meridian on each side of the corridor of the +2.50 Tact lens. The contact lens mode of the Humphrey was used to obtain the smaller measurement beam matrix (3-mm diagonal separation) and precision was set to 0.01 D. The mean difference (Humphrey minus Rotlex) between the 2 measuring devices across the 17 measurements was 0.039 ± 0.09 D. We conclude the 2 instruments provide similar astigmatic power measurements in the astigmatic areas of lenses with progressive power. This validates use of measurements made with the Rotlex instrument in the analyses performed subsequently. It should also be noted that the axis of the cylinder to the right of the corridor ranged from 139 to 145 and to the left of the corridor from 38 to 47. This confirms the prediction in Appendix A that the cylinder axes are oblique lateral to the corridor.

**Experiment 1: Tact Zone Width and Minkwitz Prediction**

The relationship between power rate in the middle of the corridor to the orthogonal 0.50 DC zone width was investigated for each of 7 left Tact lenses (+1.00 to +2.50 adds in 0.25-D steps) acquired from a Hoya Optical Laboratory. The corridor was analyzed from 1 mm below the point of minimum power in the top of the lens to 1 mm above the point of maximum power in the bottom of the lens. Points 1 mm below and above the minimum and maximum powers were used because power rate calculations included the points above and below the point for which the power rate was calculated. As result, power rate calculations for the points of maximum and minimum power could be zero or even negative, resulting in division by zero for the Minkwitz calculations and skewing data analysis.

The relationship between power rate and 0.50 DC zone width for a sample lens (+1.75 add) is shown in Figure 4. First, it can be seen that the power rate is greatest in the middle of the lens and decreases toward both the top and bottom of the lens. Data are shown in comparison to the Minkwitz theorem-calculated 0.50 DC zone width. In the middle of the lens, it may be seen that the measured 0.50 DC zone is wider than predicted by the Minkwitz theorem, and in the top and bottom of the lens of the zone, it is narrower than predicted. For each 1-mm Y increment, the difference between the measured zone width and the Minkwitz theorem-predicted zone width (based on the power rate at that location) was calculated. Along the entire length of the vertex line, the average difference (± standard deviation) for the +1.75 lens is −0.9 ± 5.9 mm. Although some regions of the corridor have widths that are greater or less than the Minkwitz prediction, because the mean is close to zero, it appears that the Minkwitz theorem works on average along the entire vertex line. Similar calculated mean differences for all 7 lenses are shown in Table 1.

Each of the 7 lenses (adds of +1.00 to +2.50) had a pattern similar to that shown for +1.75 in Figure 4. Widths near the middle of each lens were larger than predicted by the Minkwitz theorem and narrower than predicted in the top and bottom of each lens. The narrowest 0.50 DC zone width was 5 mm, hence the 4 mm averaging of the Rotlex did not result in any invalid measurements that included an area from both sides of the umbilicus. Also, at least for the medium to higher adds (+1.50 to +2.50), the differences along the vertex line appear to cancel one another. This indicates design tradeoff, i.e., zones can be made wider than predicted in some areas of the lens, but there are compensatory areas where the zones are narrower than predicted. It is as if the Minkwitz theorem predicts the average zone width/power rate relationship over the region of power change. Individual lenses can be designed to exceed the predicted zone width in some areas, but as a result must have compensatory areas with less than predicted width.

**Experiment 2: Power Rate/Zone Width Relationship Near the Middle of Tact the Corridor**

The Tact lens was initially selected for analysis because the lens had a central region where the astigmatic contours were vertical, resulting in a vertical region with constant zone width (Fig. 3).

**TABLE 1.**

For Tact lenses of designated add power, the vertical range of power change and, averaged in 1-mm steps over that range, the mean difference in width of the 0.50 DC zone between that which was measured and that which is predicted by the Minkwitz theorem

<table>
<thead>
<tr>
<th>Lens</th>
<th>Power change Range (mm)</th>
<th>Mean width difference (measured - Minkwitz in mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>+1.00</td>
<td>37</td>
<td>6.1 ± 5.6</td>
</tr>
<tr>
<td>+1.25</td>
<td>37</td>
<td>3.8 ± 7.3</td>
</tr>
<tr>
<td>+1.50</td>
<td>34</td>
<td>−1.2 ± 4.9</td>
</tr>
<tr>
<td>+1.75</td>
<td>33</td>
<td>−0.9 ± 5.9</td>
</tr>
<tr>
<td>+2.00</td>
<td>36</td>
<td>1.2 ± 3.4</td>
</tr>
<tr>
<td>+2.25</td>
<td>35</td>
<td>0.6 ± 2.3</td>
</tr>
<tr>
<td>+2.50</td>
<td>37</td>
<td>−2.0 ± 8.4</td>
</tr>
</tbody>
</table>
initially hypothesized that the Minkwitz relationship would manifest in this region of constant zone width. The results of experiment 1, however, show that the zone width in this region exceeds the prediction by the Minkwitz theorem. The purpose of experiment 2 is to further investigate the power rate/zone width relationship in this region of the lens.

The astigmatic contour plot of each of the 7 Tact lenses (adds 1.00 to +2.50) was visually inspected to identify the region of Y levels in which it appeared that the lens had relatively vertical astigmatism contours and hence constant zone width. The selected ranges are shown in Table 2. For each lens, this region was near the center of symmetry for the lens and also contained the Y level with the greatest power rate along the corridor.

For each selected region, the mean power rate within the region and the mean zone widths for each 0.25 DC increment of astigmatism were determined. These are displayed in Figure 5 along with the zone width predicted by the Minkwitz theorem. The power-rate/zone-width relationships of the lenses are very similar in shape to those predicted by the Minkwitz theorem, but in all cases, the measured widths are greater than predicted, even for the 0.25 DC zone width. The amount by which the measurements exceed the Minkwitz theorem can be seen to increase with larger magnitude of cylinder.

Using multiple regression approaches, an equation for the data shown in Figure 5 was determined. The experimental zone width (ZW) in millimeters was regressed on cylinder (cyl) in diopters, the quadratic form of cylinder, power rate (PR) in diopters/mm, and the interaction of power rate and cylinder (pr * cyl). A number of different polynomial equations were attempted and the best fitting model, as defined by $R^2$, was chosen. The best fitting equation ($R^2$ value is 0.98) through the Hoya Tact data points is:

$$ZW = 5.66 + 24.1 \text{ cyl} - 113.54 \text{ PR} + 9.13 \text{ cyl}^2 + 694 \text{ PR}^2 - 192.4 \text{ cyl} \ast \text{PR}$$

This equation is graphed along with the measured data in Figure 6.

**Experiment 3: General-Use Progressive Addition Lenses and Minkwitz Prediction**

Experiments 1 and 2 demonstrated that zone width in the Hoya Tact lens can exceed the width predicted by the Minkwitz theorem. This occurs in the middle section of the vertex line, and a polynomial equation describes the zone width as function of power rate and astigmatism level. In the areas above and below the central region, the zone widths were narrower than predicted by the Minkwitz theorem, but in all cases, the measured widths are greater than predicted, even for the 0.25 DC zone width. The amount by which the measurements exceed the Minkwitz theorem can be seen to increase with larger magnitude of cylinder.

Eight general-use PALs, listed in Table 3, were selected from the lenses used in a previous study. PALs were acquired from optical laboratories and ordered to plano distance power with +2.00 D add. The corridor range analyzed for each PAL is listed in Table 3. The zone width data (0.50 DC) for one of the 8 lenses are plotted in Figure 7, along with the Minkwitz prediction and the polynomial equation derived in experiment 2. For the sample lens in Figure 7 and for each of the other 7 PALs, the Minkwitz theorem-predicted zone width appears to bisect the measured zone widths, showing a tradeoff similar to that shown in experiment 1. For each of the 8 lenses, the width in the top portion of the lens exceeded the Minkwitz theorem prediction, whereas the width in the bottom of the lens is less than predicted. Effectively, this results in wider distance viewing zones at the expense of near viewing zone width.

The tradeoff in measured versus predicted zone width along the corridor was tested further on each lens by subtracting the Minkwitz theorem prediction for each 1-mm increment along the cor-

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**TABLE 2.**

Regions of each Hoya Tact lens selected for parallel astigmatic contours

<table>
<thead>
<tr>
<th>Lens add power (D)</th>
<th>2.50</th>
<th>2.25</th>
<th>2.00</th>
<th>1.75</th>
<th>1.50</th>
<th>1.25</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>Upper Y level (mm)</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>Lower Y level (mm)</td>
<td>7</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>9</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>
The mean difference between the measured width and the Minkwitz theorem prediction as measured for the entire length of the corridor for each lens is close to zero. As for the Hoya Tact analysis, the Minkwitz theorem prediction is approximated on average along the corridor length, although large variances from the Minkwitz theorem prediction occur at individual locations along the corridor. On average, the Minkwitz theorem prediction seems to work. It can also be noted in Figure 7 that the polynomial equation, derived based on measurements of Tact lenses, closely predicts the zone width as function of power rate and astigmatism level. In the areas above and below the central region zone, widths were narrower than predicted by the Minkwitz theorem. When averaged along the entire vertex line, however, the Minkwitz theorem prediction seems to apply.

The Hoya Tact lens was selected because of its unique features; however, experiment 3 showed similar results on other common PAL designs. The PAL designs have zone widths in the upper portion of the lens that are greater than the Minkwitz theorem calculation, but the zone widths in the lower portions of the lens are generally narrower than the Minkwitz theorem calculation. Functionally, this results in a wide distance viewing zone and a narrower near viewing zone. Like with Hoya Tact, the zone width gains and losses to the Minkwitz theorem-calculated width generally cancel one another along the length of the corridor showing that gains to the Minkwitz theorem in one section of the corridor are compensated by losses to the Minkwitz theorem in other sections of the corridor.

The Minkwitz theorem appears to predict the average zone width/power rate relationship over the length of the corridor. Individual lenses can be designed to exceed the predicted zone width in some areas, but as a result must have compensatory areas with less-than-predicted width. The contour plots of typical PALs (Fig. 1) show that the unwanted astigmatism is “shifted” toward the lower portions of the lens, suggesting that the astigmatism can be redirected from the orthogonal orientation (relative to the vertex line) assumed in the Minkwitz theorem.

**Possible Explanations for Variance From Minkwitz**

The proof of the Minkwitz theorem (Appendix A is a simplified analysis; Appendix B gives a general proof) is limited to a line of “umbilics” (Latin for “navel”) or in German “Nabelpunkten”,¹ a word chosen because a point from which a surface curves away equally in all directions looks like a navel. For a line of umbilics, the theorem is exact in the “neighborhood” of any point on that line.

### TABLE 3.
The general purpose progressive addition lenses tested in experiment 3

<table>
<thead>
<tr>
<th>Progressive addition lens</th>
<th>Corridor section (Y range)</th>
<th>Measured width minus Minkwitz prediction (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td>AO Compact</td>
<td>-7 to 16</td>
<td>-0.16</td>
</tr>
<tr>
<td>Essilor Comfort</td>
<td>-4 to 18</td>
<td>1.48</td>
</tr>
<tr>
<td>Essilor Panamic</td>
<td>-4 to 18</td>
<td>-2.29</td>
</tr>
<tr>
<td>Hoya ECP</td>
<td>-8 to 23</td>
<td>1.78</td>
</tr>
<tr>
<td>J&amp;J Definity</td>
<td>-4 to 18</td>
<td>3.68</td>
</tr>
<tr>
<td>Shamir Genesis</td>
<td>-4 to 18</td>
<td>0.54</td>
</tr>
<tr>
<td>SOLA VIP</td>
<td>-4 to 18</td>
<td>0.47</td>
</tr>
<tr>
<td>Zeiss Gradal Top</td>
<td>-6 to 20</td>
<td>1.94</td>
</tr>
</tbody>
</table>

For the indicated corridor length (chosen to include the entire power change range minus 1 mm on each end) in 1-mm increments, the mean and standard deviation of measured zone width minus Minkwitz-predicted and polynomial-predicted zone width.

**FIGURE 7.**
The relationship between zone width (0.50 DC) and power rate at 1-mm increments along the extent of the power vertex line for Essilor Comfort, one of 8 general-purpose PAL designs for which such data were obtained. The line connects contiguous locations along the vertex line. The Minkwitz relationship and polynomial equation derived in experiment 2 are also shown.

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1. Minkwitz theorem appears to predict the average zone width/power rate relationship over the length of the corridor. Individual lenses can be designed to exceed the predicted zone width in some areas, but as a result must have compensatory areas with less-than-predicted width. The contour plots of typical PALs (Fig. 1) show that the unwanted astigmatism is “shifted” toward the lower portions of the lens, suggesting that the astigmatism can be redirected from the orthogonal orientation (relative to the vertex line) assumed in the Minkwitz theorem.
The rate of change of the power can vary in any way and not affect the result because it is only the local rate of mean curvature change that matters. The Minkwitz theorem is only necessarily true in the immediate neighborhood of the umbilic line. Off of the umbilic line, the curvatures may develop at different rates and so it is possible that the net effect as measured at a distance from the umbilic does not follow the Minkwitz theorem.

Visualize a series of points parallel to and some small distance from the umbilic thereby creating a new line on the surface. This line cannot be an umbilic because oblique astigmatism has been generated in the intervening distance from the umbilic to the new line. In addition, if the mean curvature was changing along the umbilic, then the oblique astigmatism will have a different value at each point on the new line. Therefore, one finds a rate of change of not only mean curvature along the new line, but also a rate of change of astigmatism. These conditions must affect the rate of change of curvature at locations further from the umbilic. Therefore, the magnitudes of astigmatism at points distant from the center of the corridor such as measured in this study do not purely relate to the Minkwitz theorem because of the continuous alteration of the power rate/astigmatism relationship with greater distance from the umbilic.

Variance from the Minkwitz relationship appears to occur at a very small distance from the umbilic. We measured variances from the Minkwitz prediction for astigmatism values as low as 0.25 DC and as close to the corridor as the 4-mm aperture on our measurement device would allow, i.e., within 2 mm of the center of the corridor. Further study of the exact relationship between power rate and astigmatism in the area immediately surrounding the umbilic will likely require precise profile measurements.

CONCLUSIONS

The Minkwitz theorem is derived for a line of umbilics (i.e., with no astigmatism along the line) and should precisely apply to the immediate neighborhood of points surrounding the umbilic. However, there can be several local effects that can potentially modify the power rate/astigmatism relationship predicted by the Minkwitz theorem at some distance from the umbilic. These include the fact that the oblique astigmatism that builds lateral to the umbilic affects the rate at which astigmatism builds at incrementally further distances. Also, when the power rate changes along the length of the umbilic, it results in an astigmatism gradient along a line that is parallel to the umbilic. This astigmatism gradient may also influence the local buildup of astigmatism at locations somewhat lateral to the umbilic. Furthermore, it is likely that the existence of astigmatism along the vertex line in the middle of the corridor has further effects on the buildup of astigmatism lateral to the vertex line and accounts for deviation from the Minkwitz theorem.

Although the Minkwitz theorem should apply exactly to the immediate locale of an umbilic, these factors enable enough local deviation from the Minkwitz theorem to "steer" the astigmatism and affect its magnitude in the peripheral portions of a lens.

The results of this study show that deviations from the Minkwitz relationship can occur very near to the corridor. However, although the Minkwitz relationship may be altered in some regions of the corridor, there is a global component to the Minkwitz prediction that applies to PALs. On a global level for a PAL, the gains and losses of astigmatism along the corridor with respect to the Minkwitz prediction have strong tendency to cancel one another. In the end, it appears the unwanted astigmatism associated with a given power change along a given distance can be redistributed but probably not reduced.

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APPENDIX A

The appendix is available online at www.optvissci.com.

A BASIC PROGRESSIVE ADDITION LENS AND ITS APPLICATION TO THE MINKWITZ THEOREM

To demonstrate the relationship between the spherical power progression rate along a power vertex line and the astigmatism progression rate lateral to the vertex, the following characteristics of a "basic PAL" are assumed: the back surface is flat, the upper half of the lens is without power (plano), in the lower half of the lens, the positive power in vertical meridian is proportional to distance below center (y), there is no astigmatism along the vertex line (x = 0), and positive power in the horizontal meridian is independent of horizontal position (x). It is also assumed that along the vertex line, there is no discontinuity at the lens center in the first and second derivatives of the lens thickness (or surface elevation).

Consider first the lens characteristics along the vertex line. If the power on this line (which is in the vertical meridian) is proportional to distance below center, then

\[ P_y = gy \ (A1) \]

where \( g \) is a constant.

The power gradient along the vertex line will therefore be

\[ \frac{\partial P_y}{\partial y} = g \]

But the power along the vertex line is given by

\[ P_y = -(n-1)\frac{\partial^2 S_1}{\partial y^2} \]

where \( S \) is the surface elevation (or thickness).

Thus, the power gradient along the vertex line will be

\[ \frac{\partial P_y}{\partial y} = - (n-1) \frac{\partial S_1}{\partial y} = g \]

The solution of this equation is

\[ S_1 = S_0 - \frac{g}{2(n-1)}(n-1) = S_0 - k y^3 \]

where \( S_0 \) is surface elevation of the upper half and \( k = g/(6(n-1)) \). Now consider the lens characterstics at any position. If power at any point in the horizontal meridian is same as in the vertical meridian, then, in accordance with equation A1,

\[ P_x = gy. \] (A3)

This power is given by the standard equation

\[ P_x = - (n-1) \frac{\partial^2 S_1}{\partial x^2} = g y \]

The solution of this equation is

\[ S = S_1 - \frac{g}{2(n-1)}(n-1) = S_1 - 3k x^2 y \]

where \( S_1 \) as before, is the thickness along the vertex line. Substituting the value of \( S_1 \) from equation A2 gives the overall equation for surface elevation of a basic PAL

\[ S = S_0 - k y^3 - 3k x^2 y \]
This is the same formula as for the Alvarez variable power lens.  

ASTIGMATISM OF THE BASIC PAL AND THE MINKWITZ RELATIONSHIP

The horizontal-vertical astigmatism is given by

\[ P_y - P_x = -(n-1)(\partial^2 S/\partial y^2 - \partial^2 S/\partial x^2) = -(n-1)(6ky-6ky) = 0 \]

The diagonal astigmatism is given by

\[ -(n-1)2 \partial^2 S/\partial x \partial y = 12(n-1)kx = 2gx \]

Thus, astigmatism changes in the horizontal direction at a rate 2g, which is twice the vertical gradient of power (equations A1 and A3)—this is the Minkwitz relationship.

The lens powers along the principal meridians are shown in Figure A1. Note that the principal meridians are at axes 45 and 135. Along the 45 and 135 diagonals, the lens acts as a pure cylindrical lens, i.e., there is no power along the other principle meridian. In Figure A2, the power at any position is represented by the sum of mean spherical power and the unwanted crosscylinder power (the diagonal lines represent the axis and power of the converging meridian). It is seen that, in this basic PAL, mean spherical power increases in proportion to distance below the midline but is independent of horizontal position; the crosscylinder power increases in proportion to distance from the vertex line but is independent of vertical position below the midline.

APPENDIX B

A GENERAL DERIVATION OF THE MINKWITZ THEOREM

To find the basic relationship between the power progression rate, \( R \), along a power vertex line that has no astigmatism and the magnitude of unwanted astigmatism lateral to that line the following derivation is given. Such a power vertex line that does not have astigmatism is known as an umbilic line in the theory of surfaces.

The front surface of a lens with such a power vertex line is constructed so that along the power vertex line, the principal surface power values are equal but their values change at some rate as one progresses along the line, i.e., there is a progressive change of spherical power. The principal surface powers at any point on this surface are given by the familiar formula:

\[ F_i = (n-1)K_i \]

where \( F_i \) is one of the principal powers at the point

\( K_i \) is one of the principal surface curvatures at the point

It is the surface curvatures that will be considered in this treatment because the surface power values may be immediately found from them using the above relationships. The derivation begins by describing the surface elevation at any point as a function of \( x \) and \( y \), i.e., \( S(x,y) = z \). In addition, we specify that the second derivatives of the surface are continuous in the area immediately surrounding the umbilic line. It is always possible to orient such a surface so that at any given point, the \( z \) coordinate axis is normal to the surface and the \( y \) axis is directed along the umbilic line with the \( x \) axis at right angles to that line. When this is done, it is true that the curvature values are solely functions of the second derivatives of function \( S \). Described in such a coordinate system, the mean curvature of a surface, \( M \), is given by

\[ M = \frac{1}{2} \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) \]

The component of the astigmatism oriented with the \( x \) and \( y \) axes, \( C_x \), is given by

\[ C_x = \frac{\partial^2 S}{\partial x \partial y} \]

and the component of the astigmatism oriented at 45° to the \( x \) and \( y \) axes, \( C_y \), is given by

\[ C_y = \frac{\partial^2 S}{\partial y \partial x} \]

On the power vertex line where the total astigmatism is defined as being zero, we have

\[ C_x = 2\frac{\partial^2 S}{\partial x \partial y} = 0 \]

and

\[ C_y = 2\frac{\partial^2 S}{\partial y \partial x} = 0 \]

Thus, the expression for the mean curvature, \( M \), may be written with the use of (B1) and (B4) as

\[ M = \frac{1}{2} \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) \]

We now say that at a point on the umbilic line, the mean power changes at rate \( R \) in the direction of that line so that

\[ \frac{\partial M}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = R \]

Then using the expression found in (B3) for \( \frac{\partial}{\partial x} \partial^2 S \) we find that

\[ \frac{\partial}{\partial x} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = R \]

We next use the fact that the astigmatism is zero at all points on the power vertex line. Therefore, its components do not change their values as one moves along the line. Thus

\[ \frac{\partial C_x}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = 0 \]

and

\[ \frac{\partial C_y}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial y \partial x} \right) = 0 \]

so that

\[ \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial y \partial x} \right) = 0 \]

To find the rate of change of the mean curvature, \( M \), in the \( x \) direction, we first note that we could also use (B1) and (B4) to write the expression for \( M \) as

\[ M = \frac{1}{2} \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right) \]

Then, using (B8) and (B9), the rate of change of the mean curvature, \( M \), in the \( x \) direction may be written as

\[ \frac{\partial M}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial^2 S}{\partial x \partial y} \right) = 0 \]

Therefore, in the immediately surrounding area of the umbilic line, the mean curvature does not change with an initial movement.
at right angles to that line. Using this result but using (B6) to express \( M \), we find that

\[
\frac{\partial M}{\partial x} = \frac{\partial }{\partial x} \left( \frac{\partial^2 S}{\partial x^2} \right) = 0
\]

Now the rate of change of the other component of astigmatism, \( C_+ \), in the \( x \) direction may be found by using\(^4\) as follows

\[
\frac{\partial C_+}{\partial x} = \frac{\partial }{\partial x} \left( \frac{\partial^2 S}{\partial x \partial y} - \frac{\partial^2 S}{\partial y^2} \right) - \frac{\partial }{\partial x} \left( \frac{\partial^2 S}{\partial y^2} \right)
\]

Equations (9) and (11) tell us that both parts of the final expression for \( \frac{\partial C_+}{\partial x} \) are zero, so we may conclude that

\[
\frac{\partial C_+}{\partial x} = 0
\]

Thus, we arrive at the final result that as we move at right angles to the umbilic line, in the immediately surrounding area, there is an increase in oblique astigmatism, \( C_x \), equal to twice the rate, \( R \) (B7), that the mean curvature is changing along the power vertex line and that this is the total astigmatism generated as the change in the other component, \( C_+ \), is zero (B12). In addition, there is no initial change in mean curvature, \( M \), as we move at right angles to the umbilic line (B11). Figure A2.

**FIGURE A2.**
Analysis of the power of a basic progressive addition lens (PAL) in terms of contributions from mean spherical equivalent and crossed-cylinder components. Mean spherical equivalent power of the basic PAL is represented by the diameter of the circles; the diagonal lines represent the axis and power of the converging meridian of the crossed cylinder. The lens center is marked by a dot.

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