I Like Your GRIN: Design Methods for Gradient-Index Progressive Addition Lenses
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ABSTRACT
Progressive addition lenses (PALs) are vision correction lenses with a continuous change in power, used to treat the physical condition presbyopia. These lenses are currently fabricated using non-rotationally symmetric surfaces to achieve the focal power transition and aberration control. In this research, we consider the use of Gradient-Index (GRIN) designs for providing both power progression and aberration control. The use of B-Spline curves for GRIN representation is explained. Design methods and simulation results for GRIN PALs are presented. Possible uses for the design methods with other lenses, such as unifocal lenses and axicons, are also discussed.

Keywords: gradient-index lens, GRIN, progressive addition, ophthalmic, design

1. INTRODUCTION
A progressive addition lens (PAL) is a multifocal lens meant to give the presbyope (someone with presbyopia) clear vision at a range of viewing distances. Progressive addition lens design has been focused on the use of aspheric lenses, utilizing the shape of the lens' front and back surfaces, thickness, and index of refraction. Typically, the front surface is the progressive surface, which is not rotationally symmetric, and provides the power addition. The power addition is located in the progressive zone, between the distance and the near regions of the lens. A power-gradient example for a typical PAL is shown in Figure 1.1.1

![Figure 1.1. Example power progression along vertical axis of PAL](image)

Here the design of PALs via a gradient-index of refraction is examined. There have been minimal studies of gradient-index progressive lenses2-5, and many use the GRIN for a constant-power region, with the power progression added by aspheric surfaces. As suggested, a PAL has a two-dimensional, non-radially symmetric power form, and so requires a non-radially symmetric index of refraction.6-8 The progressive lens should be designed to achieve the desired power progression, while reducing the aberrations to an acceptable level. The optical aberrations of foremost concern in the initial design are mean oblique error and oblique astigmatism.6 It is generally understood that these aberrations should be less than \( \frac{1}{2} \)-diopter to be acceptable for most people.

2. PHYSICAL MODEL DESCRIPTION
To characterize a PAL, its performance is evaluated over the entire range of gaze angles for which the lens is designed. The geometry used is illustrated in Figure 1.29 The eye’s center of rotation is at Q’, a distance of q’ from the back surface, with the pupil at p’ along the gaze angle. The lens thickness is \( t \), and has a base index of refraction \( n_o \). The two
gaze angles are \( \alpha \) (vertical) and \( \beta \) (horizontal). An arbitrary ray passing through \( Q' \) can be described by these two gaze angles. The course of the power variation is along the vertical axis, called the “Main Meridian.”

![Diagram of gaze angles](image1)

Figure 1.2. (Left) Schema for ray-tracing 2D ophthalmic lenses. (Right) Geometry for tracing tangential and sagittal rays, relative to some ray along the gaze angle. Separation of tangential and sagittal rays from gaze ray are exaggerated; they are actually infinitesimally distanced from the ray.

An important aspect of the ray-tracing geometry is that the coordinate system of the rays rotates as the gaze angle traverses the lens. The tangential ray falls on a transformed vertical axis defined by the line containing the lens vertex and the ray intersection with the lens’ back surface. So, for a gaze angle intersecting the vertical axis, the tangential ray is offset vertically and the sagittal ray offset horizontally. For a ray intersecting the horizontal axis, the tangential ray is offset horizontally and the sagittal ray offset vertically. This is also illustrated in Figure 1.2. Note that when performing ray-tracing simulations, the tangential and sagittal rays are infinitesimally distanced from the ray; their distances are exaggerated in the figure.

### 3. INDEX PREDICTION AND REPRESENTATION METHODS

The necessary index profile is derived using geometrical optics and an optical path length argument. (See Figure 1.3) The incident light is assumed to be collimated and co-linear with the optical axis. Further assuming a thin sample (or equivalently, a weak gradient), the index variation experienced by any ray is small, and there is negligible beam displacement in the material, allowing the radial ray position at the emerging wavefront to be considered equal to the input position. This is a good assumption until the wavefront displacement from the back surface is no longer negligible compared with the back focal length, as shown by equation (1.1). The wavefront displacement \( z(r) \) from the back surface describes the emergent wavefront.

![Diagram of index prediction](image2)

Figure 1.3. Parameters used for deriving the index of refraction for a gradient-index concentric varifocal lens.
The slope of the wavefront normal is then given by $-dz/dr$. Because the geometrical ray angle is equal to slope of the wavefront normal $\theta(r)$, the wavefront is described by

$$\tan(\theta) = -\frac{dz}{dr} = -r \cdot \phi(r),$$

where $z(r) \ll \frac{1}{\phi(r)}$, \hspace{1cm} (1.1)

where the wavefront offset from the back surface is assumed to be much smaller than the back focal distance. The wavefront shape can also be described in terms of the local index difference from the index of refraction at the center of the lens, $n(0)$, by the equation $z(r) = t \cdot (n(0) - n(r))$. Next, consider two rays offset by the infinitesimal distance $\Delta r$ along the radius, entering the lens at radial heights $r$ and $r' = r - \Delta r$. The intersection distance $d$ from the back surface is calculated by

$$r + \theta(r)d = r' + \theta(r')d,$$

where, from equation (1.1),

$$\theta(r) = t \frac{dn(r)}{dr}.$$ \hspace{1cm} (1.3)

Expanding and rearranging (1.2), gives

$$\lim_{\Delta r \to 0} \frac{1}{d} = -\frac{t}{\Delta r}\left[\frac{dn(r)}{dr} - \frac{dn(r-\Delta r)}{dr}\right].$$ \hspace{1cm} (1.4)

Writing the derivatives in terms of $\Delta r$ and simplifying gives

$$\lim_{\Delta r \to 0} \frac{1}{d} = -\frac{t}{\Delta r}\frac{d^2n(r)}{dr^2}.$$ \hspace{1cm} (1.5)

The tangential power, equal to $1/d$, is given in terms of the index equation

$$\phi_t(r) = -t \frac{d^2n(r)}{dr^2}.$$ \hspace{1cm} (1.6)

The tangential power computation can be inverted, and the index of refraction determined if the tangential power is known,

$$n(r) = -\frac{1}{t} \sqrt{\int_0^r \phi_t(r')dr'^2}.$$ \hspace{1cm} (1.7)

Assuming eq. (1.6) is valid and separable over the entire aperture,

$$\phi_s = -t \frac{d^2n(x,y)}{dx^2} \quad \text{and} \quad \phi_s = -t \frac{d^2n(x,y)}{dy^2}.$$ \hspace{1cm} (1.8)

Equivalently, the refractive index is computed as

$$n_s(x,y) = -\frac{1}{t} \sqrt{\int_0^x \phi_s(x',y')dx'^2}, \quad n_s(x,y) = -\frac{1}{t} \sqrt{\int_0^y \phi_s(x',y')dy'^2},$$

and

$$n(x,y) = n_s(x,y) + n_s(x,y).$$ \hspace{1cm} (1.9)

To achieve the desired effective power, both the tangential and sagittal power must follow the power progression along the main meridian. As with the tangential refractive index, a complementary refractive index profile must also be determined for the sagittal power. Fortunately, an initial estimation is easily achieved using the index prediction methods and by mimicking the geometric methods used in some aspheric PAL designs. The two primary optical
aberrations considered are oblique astigmatism and mean optical error. These are eliminated if the sagittal and tangential powers are identical and equal to the desired power. For any given y-position, the sagittal power along the x-direction should be equal to the tangential power at $x=0$. From eq. (1.9), the refractive index for the x-axis is calculated as,

$$n_x(x, y) = -\frac{1}{2t} x^2 \phi_t(x) .$$

(1.10)

### 4. EXAMPLE 1: SMALL GAZE ANGLES

An equation inspired by US Patent 5,042,936 is used for the power progression along the lens’s main meridian,

$$\phi_t(y) = \frac{\phi_A}{1 + e^{k(y - \gamma)}} .$$

(1.11)

An example is shown in Figure 1.4, using the values of $\phi_A$ equal to 1 diopter (dpt), $\kappa$ equal to 0.60 mm$^{-1}$, $\gamma_S$ equal to -5 mm, and a 16 mm semi-aperture for a gaze-angle range of $\pm 30^\circ$ along the main-meridian. Typical values for the lens geometry are assumed: $Q$ equal to 27 mm, and $t$ equal to 3 mm. The predicted refractive index profile, given by eq. (1.12), is also shown in Figure 1.4.

$$n_t(y) = n_o - \frac{1}{2t} \phi_A \left[ y^2 - \frac{2}{k^2} L_{i_1} \left( -e^{k(y - \gamma)} \right) \right]$$

(1.12)

where $L_{i_1}(z) = \sum_{|n|} z^n / k^n$.

Ignoring the sagittal power for the moment, an axially symmetric (about the y-axis) 2D power profile using this power form is used. To continue with lens simulations, the base lens must be defined. A one diopter, spherical lens with minimal mean oblique error (power error) and mean oblique astigmatism is used. The base index $n_o$ is 1.5, the thickness $t$ is 3 mm, the center of rotation $q'$ is at 27 mm and the stop $p'$ is located at 14 mm. The lens’ radii of curvature are 73.32 mm and 84.56 mm for the front and back surfaces, respectively. The simulated lens performance is shown in Figure 1.5.

The one diopter lens is given the refractive index profile designed for a one diopter (tangential) power addition, and the lens simulated in same geometry as the initial lens. The tangential power along the main meridian follows the desired
power until about the 20° gaze angle, where it begins to significantly roll off. This is not surprising, the paraxial assumptions are being violated. This can be corrected by modifying the index design iteratively. The sagittal power is constant along the main meridian (except for roll-off effects), because the current refractive index has no lateral variation.

The difference between the desired power and the tangential power is computed, and fit to a polynomial (eighth order in this particular case). That equation for the power difference is integrated, per eq. (1.9), to estimate the refractive index error. The new refractive index estimate along the main meridian is the previous estimate plus one-half of the computed error (Only one-half the error amount to prevent the iteration from becoming numerically unstable.) The result of the first iteration is shown in Figure 1.6. After eight iterations, the tangential power follows the desired power curve closely, also as shown in Figure 1.6. Another benefit to the iterative process is that the resultant refractive index profile is still represented analytically.

The lens behavior is seen to be even more complex when the full lens aperture is examined, as shown in Figure 1.7. The figures are contour plots of the optical power, within the plane of the lens’ coordinate system. The upper half of each contour plot corresponds to the upper half of the lens, where the power is mostly constant. The lower half corresponds to the lower half of the lens, where the power progressive is located. The curves are isoclines of power in diopters. The perimeter within the frame of the plot which encompasses the contour lines indicates the usable aperture of the lens. Outside of that boundary, rays do not pass through the lens, but are lost due to total internal reflection.

Of particular interest in these contour plots is that the sagittal power varies as the gaze angle moves away from the vertical axis. This variation, despite the lack of horizontal variation in the refractive index is related to the ray-tracing geometry described previously, illustrated in Figure 1.2. The sagittal ray is not displaced solely from the horizontal of the gaze ray. Rather, it is along a diagonal relative to the index profile. This behavior can be predicted, as explained next.

![Figure 1.6](image1.png)  
Figure 1.6. (Left) Simulated lens performance along main meridian. Note power roll-off for tangential power near 20° gaze angle. (Right) Ray-trace results after eight iterations, correcting the tangential power roll-off.

![Figure 1.7](image2.png)  
Figure 1.7. Contour plot of (Left) tangential power and (Right) sagittal power from lens simulation. Labeled curves are optical power isoclines in diopters.
5. REFRACTIVE INDEX COORDINATE TRANSFORM

Ophthalmic lens literature often uses plots of surface power to aid in the assessment of spherical and aspherical lens performance and quality. An analogous approach can be taken with GRIN PALs, by estimating lens performance from the index of refraction. The approach is to calculate the tangential and sagittal power of the lens according to the refractive index only, without consideration to surface shape or ray-tracing simulations.

As mentioned, ray-tracing requires a rotated coordinate system according to the gaze-angle. Though a lens’ behavior could be estimated using eq. (1.8), this does not account for the rotated coordinate system of the tangential and sagittal rays. To improve the estimate, a coordinate transform is applied to the position variables. To compute the optical powers in the geometry as used for simulating lens behavior, derivatives of the representative function are needed along the sagittal and tangential axes. The index derivatives are aligned with tangential & sagittal axes by a coordinate transform, shown in Figure 1.8. The transform point is placed along the new y' axis, and substituted back into the coordinate-transform equations, yield the new coordinate system variables in terms of the non-transformed x, y coordinates, given in eq. (1.13).

![Figure 1.8. Geometry for tan/sag coordinate transform.](image)

\[
(x', y') = (0, y') = \left(0, \sqrt{x^2 + y^2}\right), \quad \tan(\theta) = x/y. \tag{1.13}
\]

The representative equations for the index profile are expanded in terms of x and y, differentiated and eq. (1.13) substituted. Using the resultant equations, the predicted behavior of the current PAL is shown in Figure 1.9. These contour plots depict the same geometry as the previous two plots and show the lines of constant optical power. Comparing Figure 1.7 to Figure 1.9, the “indicial” estimation qualitatively predicts the lens’ performance. However, the roll-off effect is not estimated. Thus, the variation of the sagittal power away from the main meridian is due to the gaze-angle geometry, despite the lack of horizontal variation in the refractive index profile.

![Figure 1.9. (Left) The tangential power addition is estimated using the tangential coordinate transform and refractive index function. (Right) The sagittal power likewise estimated. This for the initial design, ignoring sagittal power concerns. Labeled curves are optical power isoclines in diopters.](image)
6. LATERAL VARIATION IN REFRACTIVE INDEX PROFILE

The desired sagittal power has not yet been addressed. An initial estimation of the required refractive index is determined from equations (1.10), (1.11), and (1.12), yielding eq. (1.14). This expresses the lateral variation in the refractive index needed to provide the sagittal power along the main meridian. The resultant index profile is shown in Figure 1.10

\[
n(x, y) = n_0 - \frac{1}{2r} \phi_A \left[ \frac{\phi_A x^2}{1 + e^{\kappa (|y| - y_0)}} + y^2 - \frac{2}{\kappa^2} L_1 e^{\kappa (|y| - y_0)} \right].
\]  

(1.14)

Figure 1.10. Refractive index profile to provide both the desired tangential power and the desired sagittal power, along the main meridian. This is achieved by incorporating the necessary lateral variation in the refractive index profile into to provide the sagittal power.

Using this new refractive index profile, the lens behavior is estimated by the coordinate-transformed refractive-index calculations, with the results shown in Figure 1.11. These plots indicate there are still roll-off errors towards the aperture of the lens. However, the tangential power is behaving as previously, and the sagittal power now increases along the main meridian. The lens is also ray-traced, shown in Figure 1.12. It is seen that the behavior predicted by physical simulation is qualitatively the same as predicted by the coordinate-transformed refractive index functions. This shows that a progressive GRIN lens can be qualitatively simulated solely on the refractive-index profile, without the need for the more computationally expensive ray-tracing procedure.

Figure 1.11. (Left) The tangential power addition is estimated using the tangential coordinate transform and refractive index function. (Right) The sagittal power likewise estimated. This for the design incorporating sagittal power concerns.
The initial prediction of the two-dimensional refractive-index profile is effective, providing much of the desired behavior. But it suffers from power roll-off at higher, off-axis gaze angles. Again, this is corrected iteratively and the results after 10 iterations shown in Figure 1.13. The lens is again ray-traced to determine the performance; the results are shown in Figure 1.14.

The progressive lens has the desired power progression in the upper distance portion, along the main meridian, and in the lower near portions. The aberrations are also small in the vicinity of the main meridian. The errors do increase significantly in the middle, side areas of the lens. These errors are not addressed in the present work.
7. EXAMPLE 2: LARGE GAZE ANGLES

The design process is now repeated for a lens with larger gaze angles, ±50° along the main-meridian. The power form is again based on eq. (1.11). The parameters are \( \phi_0 \) equal to 3 diopters (dpt), \( \kappa \) equal to 0.25 mm\(^{-1} \), and \( y_S \) equal to -10 mm. The power form and predicted refractive index are shown in Figure 1.15. This index form is currently not physically realizable by any known GRIN manufacturing technique. But for the present research, this concern will be set aside, to pursue what could be done if an arbitrary index of refraction could be fabricated.

The necessary two-dimensional refractive index profile is again estimated using eq. (1.14), shown in Figure 1.16.

The tangential power and sagittal power (Figure 1.17) both behave as desired in the upper, nearly-constant power region, and in the progressive region along the main meridian and adjacent regions. The power deviates from the desired form in the lower, lateral regions. The lens behavior is similar as in the previous example.
8. CONCLUSION

A systematic approach to designing a gradient-index profile for PALs has been given. It was shown that a power progression could be achieved solely by use of gradient index of refraction, without the use of aspheric surfaces. An approach was presented for predicting the necessary refractive index profile to give both the desired tangential and sagittal optical powers. A method to estimate lens performance based on the refractive-index profile was described; these results were compared to ray-trace simulations. Finally, two examples were given, illustrating this design approach. The lens performance was as desired in the upper, distance portion of the lens, along the main meridian, and in its vicinity. The aberrations became significant in the side areas of the lens, away from the main meridian.

9. REFERENCES